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The diffraction of surface waves by plane vertical obstacles

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A short-wave asymptotic solution is derived for the problem of the diffraction of a surface wave train, in deep water, by a two-dimensional obstacle with plane vertical sides near its intersection with the free surface. Using matched asymptotic approximations, a detailed analysis is presented for the special case of a rectangular scatterer of depth a and width $2b$, and the solution is then generalized to deal with a wider class of geometries. It is found that the transmission coefficient, at small wavelengths, has an exponentially small factor that depends on the depths of the plane sides, and an algebraically small factor that depends on the corner angles.

1. Introduction and formulation

The problem of scattering of short surface waves on deep water by large obstacles has received considerable attention in the literature and has been tackled by a wide variety of methods. In the short-wave limit it has been found that the transmission and radiation properties depend crucially on the details of the obstacle geometry, particularly near its intersection with the free surface. The present work deals with the two-dimensional situation where both sides of the scatterer are plane and vertical near the surface and uses the method of matched expansions.

Earlier work has developed the method for geometries that meet the surface obliquely (Alker 1976) and curved obstacles that meet the surface at right angles (Leppington 1973). In the short-wave limit where the wavelength $2\pi\epsilon$ is small compared with a characteristic obstacle dimension a, the earlier work has been based on the idea of dividing the fluid region into two overlapping parts. In the outer region (many wavelengths from the surface) the free-surface condition is simplified by formally letting $\epsilon \to 0$ there. In the inner regions (at distances small compared with a from the intersection points) the actual scattering geometry is replaced by the *local* geometry, which leads to relatively simple wedge problems in the presence of a free surface. The wave-free parts of inner and outer approximations are then matched to complete the asymptotic solution at all points.

For the class of geometries that are locally plane and vertical near the free surface, the procedure outlined above does not succeed without some modification, and this is the problem investigated here. Essentially the difficulty is that the local inner field near the front intersection point is that of a totally reflected wave, with no residual wave-free term to match with an outer solution.

An exact solution is available (Ursell 1947) for one particular geometry of this class, namely that of a vertical barrier of zero thickness and finite depth. For this problem the transmission coefficient is exponentially small and the reflexion coefficient differs

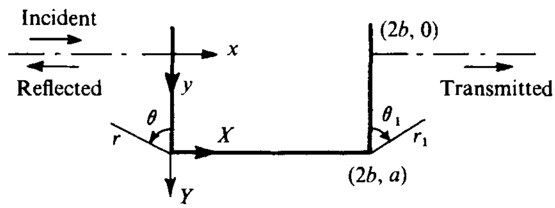

FIGURE 1. The rectangular cross-section and the co-ordinate systems.

from unity by an exponentially small term; similar properties are obviously to be expected for the more general geometries considered here. This indicates the futility of trying asymptotic expansions in increasing powers of ϵ near the intersection points, for any successful attack must account for the possibility of exponentially small scaling factors.

The key to further progress is suggested by Ursell's exact solution, which can be shown to exhibit critical behaviour near the lower tip, where the potential is found to change rapidly on a (small) wavelength scale. This suggests the presence of an inner region near the barrier tip or, more generally, near the point (or points) where the geometry changes from being plane and vertical.

For definiteness, a detailed analysis is first given (in §§ 2–4) for the prototype problem of a rectangular scatterer of depth a and breadth $2b$, and the solution is generalized in § 5.

Cartesian co-ordinates (x,y) are chosen such that the free surface is at $y = 0$ and the obstacle boundary is given by $(x = 0, y \leqslant a)$, $(0 \leqslant x \leqslant 2b, y = a)$, $(x = 2b, y \leqslant a)$ as is shown in figure 1. The velocity potential $\operatorname{Re}\{\psi(x,y)\exp(-i\omega t)\}$ is taken to be simple harmonic in time with angular frequency ω, and satisfies the two-dimensional Laplace equation for an incompressible fluid. On the free surface ψ satisfies the linearized boundary condition

$$\psi + \epsilon\,\partial\psi/\partial y = 0, \tag{1.1}$$

where $\epsilon = g/\omega^2$ and g is the acceleration due to gravity. The incident wave train has the potential

$$\psi_i \exp(-i\omega t) = \exp\{(ix-y)/\epsilon\}\exp(-i\omega t) \tag{1.2}$$

and the time factor will henceforth be suppressed. Formula (1.2) shows that the wavelength is $2\pi\epsilon$.

On the fixed scatterer the normal velocity vanishes, thus

$$\partial\psi/\partial n = 0, \tag{1.3}$$

where n is the outward normal. Uniqueness requires a further *edge condition*

$$r\,\partial\psi/\partial r \to 0 \quad \text{as} \quad r \to 0, \tag{1.4}$$

where r is the distance from either corner, and *a radiation condition* at infinity to ensure outgoing waves. Thus

$$\left.\begin{aligned} \psi &\sim \mathscr{T}\exp\{(ix-y)/\epsilon\} && \text{as} \quad x \to +\infty, \\ \psi - \psi_i &\sim \mathscr{R}\exp\{(-ix-y)/\epsilon\} && \text{as} \quad x \to -\infty, \end{aligned}\right\} \tag{1.5}$$

where the transmission and reflexion coefficients $\mathscr{T}$ and $\mathscr{R}$ are to be found. (Some authors refer to the real quantities $|\mathscr{T}|^2$ and $|\mathscr{R}|^2$ as the transmission and reflexion coefficients.)

In the short-wave limit it is expected that the incident wave will be almost totally reflected, and with this in mind it is convenient to write ψ in the form

$$\psi = \begin{cases} 2\cos(x/\epsilon)\exp(-y/\epsilon)+\phi, & x<0, \\ \phi, & x>0, \end{cases} \tag{1.6}$$

where the leading term is the potential scattered by an *infinite* plane vertical wall.

The residual function ϕ clearly has a discontinuity across the half-plane $x = 0$, $y > a$, and is specified by the conditions

$$\left.\begin{array}{c} (\partial^2/\partial x^2+\partial^2/\partial y^2)\,\phi = 0 \quad \text{in the fluid,} \\ \partial\phi/\partial n = 0 \quad \text{on the scatterer,} \\ \phi+\epsilon\,\partial\phi/\partial y = 0 \quad \text{on the free surface,} \\ [\phi] = 2\exp(-y/\epsilon), \quad [\partial\phi/\partial x] = 0, \quad y>a, \end{array}\right\} \tag{1.7}$$

where $[\phi]$ denotes the discontinuity $\phi(0+,y)-\phi(0-,y)$.

In addition ϕ satisfies edge conditions like (1.4) and the radiation conditions

$$\phi \sim \begin{cases} \mathscr{T}\exp\{(ix-y)/\epsilon\} & \text{as} \quad x\to+\infty, \\ (\mathscr{R}-1)\exp\{(-ix-y)/\epsilon\} & \text{as} \quad x\to-\infty. \end{cases} \tag{1.8}$$

Numerical results have been derived for the rectangular scatterer by Mei & Black (1969) for finite depth and various values of the geometrical parameters. The wavelength is comparable with the length a, so that there is little overlap with the asymptotic results derived here.

2. Outer approximation

The outer region consists of the whole fluid domain except for a small area within a few (small) wavelengths from the edge $(0, a)$ and a thin layer close to the free surface. An asymptotic approximation is sought by simply setting $\epsilon = 0$ in the surface condition of formulae (1.7), so that ϵ does not appear explicitly in the specifications. Thus we write

$$\phi \sim k\alpha(\epsilon)\,\phi_0(x,y) \quad \text{as} \quad \epsilon\to 0, \tag{2.1}$$

where k and $\alpha(\epsilon)$ denote a constant and a scale factor to be determined.

The function ϕ_0 is harmonic and subject to the boundary conditions

$$\phi_0 = 0 \quad \text{on the free surface} \tag{2.2}$$

and

$$\partial\phi_0/\partial n = 0 \quad \text{on the scatterer.} \tag{2.3}$$

Since the surface wave trains are absent in the outer region the condition at infinity is simply that

$$\phi_0 \to 0 \quad \text{at infinity.} \tag{2.4}$$

Now since the total potential ψ is continuous, it follows from (1.7) that ϕ has a jump discontinuity of magnitude $2e^{-y/\epsilon}$ across the half-plane $x = 0$, $y > a$. It transpires, however, that the scale factor $\alpha(\epsilon)$ of formula (2.1) has the value $\epsilon^{\frac{2}{3}} \exp(-a/\epsilon)$, and the discontinuity in ϕ is therefore of negligible magnitude in the outer region where $y-a \gg \epsilon$. Thus the leading outer potential ϕ_0 is regular across the half-plane $x = 0$, $y > a$.

The problem for ϕ_0 is evidently a homogeneous one, and therefore has a non-trivial solution only if it is singular at some point (or points). Since the original potential is regular everywhere it follows that the singular point must be outside the outer region and is therefore on the free surface or else at one (or both) of the edges $(0, a)$ or $(2b, a)$. An 'inner' approximation then plays the role of smoothing out the singular behaviour, the transition from one regime to the other being accomplished by an appropriate matching argument.

It is found that the singular point for ϕ_0 can occur only at the edge $(0, a)$ that faces the incoming wave; other singularities cannot be matched to any inner approximations. On account of the boundary condition (2.3), we can obviously anticipate that the singularity will have the form

$$\phi_0 \sim r^{-\frac{2}{3}} \cos \tfrac{2}{3}\theta \quad \text{as} \quad r \to 0 \tag{2.5}$$

in the polar co-ordinate system shown in figure 1.

Any multiplicative constant factor appropriate to the outer potential (2.1) can be accommodated by the scaling constant k.

At this stage we should not rule out the possibility of a higher-order singularity, for example $\phi_0 \sim r^{-\frac{4}{3}} \cos \frac{4}{3}\theta$. Such a singularity is rejected on the grounds that it would not match with the inner solution near the edge [cf. formula (3.22)].

The conditions (2.2)–(2.5) completely specify ϕ_0, though in practice the function can be determined only implicitly using a Schwarz–Christoffel transformation involving elliptic integrals (see the appendix). For general values of the parameter b/a the solution is complicated, but is simplified if b/a is either large or small. The function $\phi_0(x, y)$ can, however, be regarded as being known in principle. In particular, its behaviour near the other edge $(2b, a)$ has the form

$$\phi_0 \sim p + q r_1^{\frac{2}{3}} \cos \tfrac{2}{3}\theta_1, \tag{2.6}$$

where (r_1, θ_1) are polar co-ordinates based at the edge $(2b, a)$ (figure 1); the constants p and q depend on the geometrical parameter b/a, and are known in principle.

3. Inner approximation

In the vicinity of the edge $(0, a)$ we seek an inner solution that will smooth out the singularity that appears in the outer approximation. The inner region is taken as the domain

$$r \ll \min(a, b), \tag{3.1}$$

so that the local geometry appears simply as a right-angled wedge, and we expect the potential there to be relatively insensitive to the precise details of the geometry away from the edge. Inner co-ordinates based on the wavelength scale are defined by the transformation

$$x = \epsilon X, \quad y = a + \epsilon Y, \quad \phi(x, y) = \Phi(X, Y). \tag{3.2}$$

To determine the overall scale of the function Φ, note that the only inhomogeneous condition in the specifications (1.7) is the fourth formula, which becomes

$$[\Phi] = 2\exp(-Y - a/\epsilon), \quad Y > 0,$$

where $[\Phi]$ means $\Phi(0+, Y) - \Phi(0-, Y)$. This suggests that we write

$$\Phi \sim \exp(-a/\epsilon)\,\Phi_0(X, Y) \quad \text{as} \quad \epsilon \to 0, \tag{3.3}$$

where Φ_0 is harmonic. Substitution into the boundary conditions (1.7) leads to the requirements

$$\partial\Phi_0/\partial Y = 0, \quad Y = 0, \quad X > 0, \tag{3.4}$$

$$\partial\Phi_0/\partial X = 0, \quad X = 0, \quad Y < 0, \tag{3.5}$$

$$[\partial\Phi_0/\partial X] = 0, \quad Y > 0, \tag{3.6}$$

$$[\Phi_0] = 2e^{-Y}, \quad Y > 0. \tag{3.7}$$

The edge condition at $(0, 0)$ is that

$$\partial\Phi_0/\partial R = O(R^{-\frac{1}{3}}) \quad \text{as} \quad R \to 0, \tag{3.8}$$

where
$$R = (X^2 + Y^2)^{\frac{1}{2}} = r/\epsilon \tag{3.9}$$

is a polar co-ordinate based on the inner variables. A more general condition than (3.8), namely that $R\,\partial\Phi_0/\partial R \to 0$, leads to the specific behaviour (3.8), which is taken here at the outset.

Finally we need a boundary condition at infinity, and this is provided by a matching argument. The inner solution $[r \ll \min(a, b)]$ and the outer approximation $(r \gg \epsilon)$ are both required to hold in the common region $\epsilon \ll r \ll \min(a, b)$, where $r \to 0$ and $R \to \infty$ simultaneously. This overlap clearly requires that $\epsilon \ll \min(a, b)$ and this is certainly the case if we consider the geometrical parameters a and b as fixed, and let $\epsilon \to 0$. With this understanding, we have

$$\Phi(R \to \infty) \sim \phi(r \to 0) \tag{3.10}$$

as our matching requirement.

Now the outer approximation (2.1), together with (2.5), shows that

$$\begin{aligned}\phi &\sim k\alpha(\epsilon)\, r^{-\frac{2}{3}} \cos \tfrac{2}{3}\theta \\ &= k\alpha(\epsilon)\, \epsilon^{-\frac{2}{3}} R^{-\frac{2}{3}} \cos \tfrac{2}{3}\theta.\end{aligned}$$

It follows at once from (3.3) that the scale function $\alpha(\epsilon)$ is given by

$$\alpha(\epsilon) = \epsilon^{\frac{2}{3}} \exp(-a/\epsilon) \tag{3.11}$$

and that the far-field behaviour of Φ_0 is given by

$$\Phi_0 \sim kR^{-\frac{2}{3}} \cos \tfrac{2}{3}\theta \quad \text{as} \quad R \to \infty, \tag{3.12}$$

where k has to be found.

The harmonic function Φ_0 is uniquely determined by the conditions (3.4)–(3.8) and (3.12). Its overall scale is determined by the inhomogeneous term (3.7) so the constant k of (3.12) cannot be prescribed arbitrarily. The solution for Φ_0 that follows will show that k must take a specific value [given by (3.23)] and the outer approximation (2.1) is then complete.

Calculation of Φ_0

The function Φ_0 satisfies the Laplace equation

$$\left(\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial \theta^2}\right)\Phi_0 = 0, \quad 0 \leqslant \theta \leqslant \tfrac{3}{2}\pi, \tag{3.13}$$

together with the conditions (3.4)–(3.8) and (3.12), where (R, θ) are polar co-ordinates. Since (3.13) is homogeneous in R, it is natural to seek a solution by Mellin transformation. Thus we define the transform

$$\hat{\Phi}_0(s,\theta) = \int_0^\infty \Phi_0(R,\theta)\, R^{s-1}\, dR. \tag{3.14}$$

Taking note of the limiting forms of Φ_0 given by (3.8) and (3.12) for small and large values of R, it is seen that the integral (3.14) converges if

$$0 < \operatorname{Re} s < \tfrac{2}{3} \tag{3.15}$$

and that the Bromwich inversion integral for Φ_0 is

$$\Phi_0(R,\theta) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \hat{\Phi}_0(s,\theta)\, R^{-s}\, ds, \tag{3.16}$$

where the real number σ lies in the range (3.15).

Under this transformation, the Laplace equation (3.13) becomes

$$(\partial^2/\partial\theta^2 + s^2)\,\hat{\Phi}_0(s,\theta) = 0, \tag{3.17}$$

and the boundary conditions (3.4)–(4.7) require that

$$\partial\hat{\Phi}_0/\partial\theta = 0 \quad \text{when} \quad \theta = 0, \tfrac{3}{2}\pi, \tag{3.18}$$

with $\partial\hat{\Phi}_0/\partial\theta$ continuous at $\theta = \pi$, and

$$\hat{\Phi}_0(s, \pi+0) - \hat{\Phi}_0(s, \pi-0) = 2\Gamma(s), \tag{3.19}$$

where $\Gamma(s)$ is the gamma function. The solution is found to be

$$\sin(\tfrac{3}{2}s\pi)\,\hat{\Phi}_0(s,\theta) = \begin{cases} -2\Gamma(s)\sin(\tfrac{1}{2}s\pi)\cos s\theta, & 0 < \theta < \pi, \\ 2\Gamma(s)\sin s\pi \cos s(\theta - \tfrac{3}{2}\pi), & \pi < \theta < \tfrac{3}{2}\pi. \end{cases} \tag{3.20}$$

Taking the range $0 < \theta < \pi$, for example, the solution for Φ_0 is

$$\Phi_0 = \frac{i}{\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma(s)\sin(\tfrac{1}{2}s\pi)}{\sin(\tfrac{3}{2}s\pi)} \cos(s\theta)\, R^{-s}\, ds, \tag{3.21}$$

and we need its limiting behaviour at large R, in order to confirm the predicted form (3.12) and to calculate k.

The integrand of (3.21) has poles at $s = -n$ and also at $s = 2m \pm \frac{2}{3}$, where n is a non-negative integer and m is any integer. An asymptotic expansion for large R is found by deforming the vertical integration path around successive poles on the positive real axis. In particular, the leading term arises from the pole at $s = \frac{2}{3}$, thus

$$\Phi_0(R,\theta) \sim -(2/\pi)\, 3^{-\frac{1}{2}}\,\Gamma(\tfrac{2}{3})\, R^{-\frac{2}{3}} \cos \tfrac{2}{3}\theta, \tag{3.22}$$

which is of the predicted form (3.12). A similar calculation for the range $\pi < \theta < \frac{3}{2}\pi$ leads to the same asymptotic form for $\Phi_0(R,\theta)$, and a comparison of (3.22) and (3.12) shows that

$$k = -(2/\pi)\,3^{-\frac{1}{2}}\Gamma(\tfrac{2}{3}). \tag{3.23}$$

Our matched solution is now complete to leading order.

The velocity distribution $\partial\Phi_0/\partial X$ on the half-plane $X = 0$, $Y > 0$ is given by $R^{-1}(\partial\Phi_0/\partial\theta)$ evaluated at $\theta = \pi$. Thus from (3.20)

$$\frac{\partial\Phi_0}{\partial X}(0, Y) = -\frac{i}{\pi}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{s\Gamma(s)\sin s\pi \sin(\frac{1}{2}s\pi)}{\sin(\frac{3}{2}s\pi)}\, Y^{-s-1}\,ds \tag{3.24}$$

for $Y > 0$, a result that is used in §4. The integrand in (3.24) has no pole at $s = 0$ and the integration path may be shifted such that $-\frac{2}{3} < \sigma < 0$.

4. Transmission and reflexion coefficients

The analysis of §§2 and 3 gives the leading term for the potential throughout the whole fluid region. Apart from the totally reflected wave term (1.6), the residual potential has been found to order $\epsilon^{\frac{2}{3}}\exp(-a/\epsilon)$ in the outer region and to order $\exp(-a/\epsilon)$ in the inner region near $(0, a)$. To this leading order the incident wave is totally reflected, thus $\mathscr{R} \sim 1$ and $\mathscr{T} \sim 0$.

In order to improve these crude approximations, from our asymptotic solution for ϕ, Green's theorem is used to express $\mathscr{T}$ and $\mathscr{R}$ in terms of the velocity distributions on the respective half-planes $x = 2b$, $y \geqslant a$ and $x = 0$, $y \geqslant a$. The fundamental Green's function of the problem is given by

$$\begin{aligned} G(x,y;x',y') = {} & -i\exp\left\{\frac{i|x-x'|-(y+y')}{\epsilon}\right\} + \frac{1}{4\pi}\log\left\{\frac{(x-x')^2+(y-y')^2}{(x-x')^2+(y+y')^2}\right\} \\ & -\frac{1}{\pi}\int_0^\infty \frac{t\cos\{(y+y')\,t/\epsilon\} - \sin\{(y+y')\,t/\epsilon\}}{1+t^2}\exp\left\{\frac{-|x-x'|\,t}{\epsilon}\right\}dt \end{aligned} \tag{4.1}$$

(see John 1950, for example). This function satisfies

$$(\partial^2/\partial x^2 + \partial^2/\partial y^2)\,G = \delta(x-x')\,\delta(y-y'),$$

where δ is the Dirac delta function, and the free-surface condition (1.1) and the radiation condition at infinity.

It is obvious that the function $G(x,y;x',y') + G(x,y;4b-x',y')$ has the additional property of having zero x derivative on the plane $x = 2b$; thus on applying Green's formula to this function and the potential $\phi(x,y)$, in the domain $x \geqslant 2b$, $y \geqslant 0$, we are led to the identity

$$\phi(x',y') + 2\int_a^\infty G(2b,y;x',y')\frac{\partial\phi}{\partial x}(2b,y)\,dy. \tag{4.2}$$

The integration runs from a to ∞ since $\partial\phi/\partial x$ is zero for $y < a$, $x = 2b$.

In particular, when $x' \to \infty$ the function G can be replaced by the first (surface wave) term of (4.1). Thus from (4.1), (4.2) and (1.8), the transmission coefficient $\mathscr{T}$ is given exactly by

$$\mathscr{T} = -2i\exp(-2ib/\epsilon)\int_a^\infty \frac{\partial\phi}{\partial x}(2b,y)\exp(-y/\epsilon)\,dy, \tag{4.3}$$

which is valid for all values of the wavelength and the geometrical parameters a and b.

Under our short-wave approximation of §§2 and 3, the velocity distribution in (4.3) can be replaced by the estimate (2.1). Furthermore, the exponential factor in the integrand of (4.3) ensures that the leading contribution to the integral comes from the vicinity of $y = a$, by Watson's lemma. This means that we may use the edge behaviour (2.6) for ϕ_0, to get

$$\mathscr{T} \sim -2ik\alpha(\epsilon)\,q\exp(-2ib/\epsilon)\int_a^\infty 3^{-\frac{1}{2}}(y-a)^{-\frac{1}{3}}\exp(-y/\epsilon)\,dy,$$

and on performing the integration and using formulae (3.23) and (3.11) for k and $\alpha(\epsilon)$, we find

$$\mathscr{T} \sim (4i/3\pi)\{\Gamma(\tfrac{2}{3})\}^2 q\epsilon^{\frac{4}{3}}\exp\{-2(a+ib)/\epsilon\}. \tag{4.4}$$

The constant q depends on the ratio a/b and is defined implicitly by (2.6). It is shown in the appendix that q can be given explicitly for three special cases. Thus if the barrier is thin, so that $b \ll a$, then $q \sim \frac{1}{16}(3\pi/4b)^{\frac{4}{3}}$, so that

$$\mathscr{T} \sim \tfrac{1}{16}i(\tfrac{3}{4}\pi)^{\frac{1}{3}}\{\Gamma(\tfrac{2}{3})\}^2(\epsilon/b)^{\frac{4}{3}}\exp\{-2(a+ib)/\epsilon\}. \tag{4.5}$$

Note that this value of $\mathscr{T}$ for a thin barrier is *not* the same as that obtained by Ursell for a barrier of zero thickness ($b = 0$). There is no contradiction here, however, since implicit in the present work is the assumption that the wavelength is small compared with the other length scales, so that the constraint $\epsilon \ll b$ limits the validity of our result. The case where b is of order ϵ, or less, would require separate treatment.

If the barrier is a long flat one, with $b \gg a$, then formula (A 11) for q can be used in conjunction with (4.4) to show that

$$\mathscr{T} \sim (\tfrac{1}{2}i/\pi)(\tfrac{3}{4}\pi)^{\frac{1}{3}}\{\Gamma(\tfrac{2}{3})\}^2(\epsilon^{\frac{4}{3}}/ba^{\frac{1}{3}})\exp\{-2(a+ib)/\epsilon\}. \tag{4.6}$$

Finally, when the scatterer is a half-square, whence $b = a$, formulae (A 9) and (4.4) give the result

$$\mathscr{T} \sim i\pi\{\Gamma(\tfrac{2}{3})\}^2\{\Gamma(\tfrac{1}{4})\}^{-\frac{8}{3}}(3\epsilon/2a)^{\frac{4}{3}}\exp\{-2(1+i)\,a/\epsilon\}. \tag{4.7}$$

A similar analysis leads to an estimate for the reflexion constant $\mathscr{R}$. Green's formula, applied to $\phi(x,y)$ and $G(x,y;x',y')+G(x,y;\,-x',y')$ in the region $x \leqslant 0$, $y \geqslant 0$, leads to the identity

$$\phi(x',y') = -2\int_a^\infty G(0,y;\,x',y')\frac{\partial\phi}{\partial x}(0,y)\,dy,$$

whence

$$\mathscr{R}-1 = 2i\int_a^\infty \frac{\partial\phi}{\partial x}(0,y)\exp(-y/\epsilon)\,dy. \tag{4.8}$$

In this case the integration path contains points of both the outer and the inner region, but the main contribution again arises from the neighbourhood of the point $y = a$. It is found then that the leading term is obtained by using the inner approximation (3.3) and (3.24), to get

$$\begin{aligned}\mathscr{R}-1 &\sim 2i\exp(-2a/\epsilon)\int_0^\infty \frac{\partial\Phi_0}{\partial X}(0,Y)\,e^{-Y}\,dY\\ &= -4i3^{-\frac{3}{2}}\exp(-2a/\epsilon)\end{aligned} \tag{4.9}$$

on using the expression (3.24) and changing the order of integration.

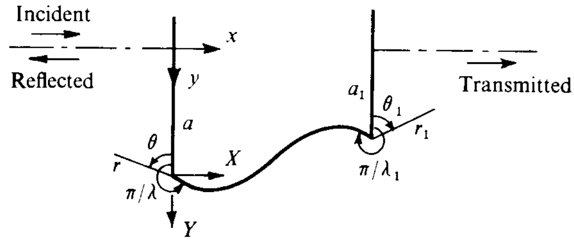

FIGURE 2. The cross-sectional profile and the co-ordinate systems.

Note that this estimate is independent of the outer solution, and in particular is insensitive to the breadth parameter b. This length scale would, of course, enter at higher-order approximations. It is worthy of note that $\mathscr{R}-1$ is imaginary to this leading order, this being consistent with (4.4) and the overall energy requirement $|\mathscr{T}|^2+|\mathscr{R}|^2 = 1$; this feature arises in any such problem when $|\mathscr{R}| \gg |\mathscr{T}|$.

5. Generalization of results

The method of the previous sections can readily be generalized to deal with a much wider class of scattering geometries, with plane vertical sides of depths a and a_1, and corner angles π/λ and π/λ_1, with $\frac{1}{2} \leqslant \lambda, \lambda_1 < 1$ (figure 2). The section $y = y(x)$ between the two corners $(0, a)$ and $(2b, a_1)$ is assumed only to be positive, continuous and to be contained within the planes $x = 0$ and $x = 2b$. There is no uniqueness theorem for this configuration, but our approximate solution *is* uniquely determined. The case $\lambda = 1$, or $\lambda_1 = 1$, needs a slightly modified treatment and will not be included here.

Proceeding as before, the incident and totally reflected waves are subtracted out by defining the residual potential ϕ of (1.6).

Outer solution

An outer approximation for ϕ, valid except near the edge $(0, a)$, is again expressed as

$$\phi \sim k\alpha(\epsilon)\,\phi_0 \quad \text{as} \quad \epsilon \to 0, \tag{5.1}$$

where ϕ_0 is harmonic and vanishes on the free surface and at infinity, having zero normal derivative on the scatterer. By analogy with the edge condition (2.5), the function ϕ_0 now has the singular edge behaviour

$$\phi_0 \sim r^{-\lambda}\cos\lambda\theta \quad \text{as} \quad r \to 0, \tag{5.2}$$

in the co-ordinate system of figure 2. This is obviously consistent with the requirement of vanishing normal derivative, and reduces to the previous value [formula (2.5)] for a corner of angle $\frac{3}{2}\pi$, with $\lambda = \frac{2}{3}$.

In principle the limit potential ϕ_0 can be regarded as known, and has the anticipated behaviour

$$\phi_0 \sim p + q r_1^{\lambda_1}\cos\lambda_1\theta_1 \tag{5.3}$$

near the other edge $(2b, a_1)$, of angle π/λ_1.

Inner solution

Defining inner variables (3.2) as before, the inner approximation is written as

$$\Phi \sim \exp(-a/\epsilon)\,\Phi_0(X, Y) \quad \text{as} \quad \epsilon \to 0, \tag{5.4}$$

where Φ_0 has to be harmonic in the wedge $0 \leqslant \theta \leqslant \pi/\lambda$, with zero normal derivative on the sides $\theta = 0$ and $\theta = \pi/\lambda$; it is subject to the discontinuity conditions (3.6) and (3.7) and an edge condition at $R = 0$.

Matching $\Phi(R \to \infty)$ with $\phi(r \to 0)$ shows that

$$\Phi_0 \sim kR^{-\lambda} \cos \lambda\theta \quad \text{as} \quad R \to \infty \tag{5.5}$$

and that the scaling function $\alpha(\epsilon)$ of formula (5.1) is

$$\alpha(\epsilon) = \epsilon^{\lambda} \exp(-(a/\epsilon). \tag{5.6}$$

A solution for Φ_0 can again be effected by Mellin transformation. It is found that

$$\sin(s\pi/\lambda)\,\Phi_0(s,\theta) = \begin{cases} -2\Gamma(s) \sin[s\pi(1-\lambda)/\lambda] \cos s\theta, & 0 \leqslant \theta < \pi, \\ 2\Gamma(s) \sin s\pi \cos s(\theta - \pi/\lambda), & \pi < \theta \leqslant \pi/\lambda, \end{cases} \tag{5.7}$$

and $\Phi_0(R,\theta)$ is given by the inversion formula (3.16) with $0 < \sigma < \lambda$. The pole contribution at $s = \lambda$ gives the leading term for large R, and this has the form (5.5) with

$$k = -(2\lambda/\pi)\,\Gamma(\lambda) \sin \lambda\pi. \tag{5.8}$$

The values (5.6) and (5.8) for α and k complete the outer solution (5.1).

Transmission coefficient

An integral expression for the transmission constant $\mathscr{T}$ is given by (4.3), with the lower limit replaced by a_1, and the leading term comes from the vicinity of the point $y = a_1$. Use of the estimates (5.1) and (5.3) then leads to the result

$$\mathscr{T} \sim (4i/\pi)\, q\lambda\lambda_1 \sin \lambda\pi \sin \lambda_1\pi\, \Gamma(\lambda)\, \Gamma(\lambda_1)\, \epsilon^{\lambda+\lambda_1} \exp\{-(a + a_1 + 2ib)/\epsilon\}. \tag{5.9}$$

This clearly reduces to the previous value (4.4) when $\lambda = \lambda_1 = \frac{2}{3}$ and $a_1 = a$.

The exponential factor $\exp\{-(a_1 + a)/\epsilon\}$ depends only on the depths a and a_1 of the vertical sides, while the algebraic ϵ-dependent factors depend only on the corner angles. Details of the global geometry of the scatterer affect only the parameter q, which is independent of ϵ and defined implicitly by (5.3).

Symmetry of solution

It is well known that the transmission constant $\mathscr{T}$ remains unchanged, for a given scatterer, if the direction of the incident wave is reversed. In the estimate (5.9), the symmetry of the formula is evident, and it remains only to verify that the coefficient q has the appropriate behaviour.

In the complementary problem, with the direction of the incident wave reversed, the outer potential ϕ_0 and the constant q would be replaced by ϕ_0^* and q^*. Here ϕ_0^* has the given singular behaviour

$$\phi_0^* \sim r_1^{-\lambda_1} \cos \lambda_1\theta_1 \quad \text{as} \quad r_1 \to 0$$

at the edge $(2b, a_1)$, and near the other edge $(0, a)$ we have

$$\phi_0^* \sim p^* + q^* r^\lambda \cos\lambda\theta \quad \text{as} \quad r \to 0,$$

this being the formula defining q^*. In order to verify the required symmetry condition, $q = q^*$, apply Green's formula to ϕ_0 and ϕ_0^* in the region outside the scatterer, excluding the singular points $(2b, a_1)$ and $(0, a)$ by small circular arcs. Because of the homogeneous boundary conditions associated with ϕ_0 and ϕ_0^*, the only contributions come from the small arcs, whence it is found that $q - q^* = 0$, as required.

Appendix. Solution of the outer potential

The method of conformal transformation is used here to solve for the potential ϕ_0 defined by (2.2)–(2.5). Using the complex variables $z = x + iy$ and $\zeta = \xi + i\eta$, our starting point is the Schwarz–Christoffel transformation

$$z = b + ia + h\int_0^\zeta \left(\frac{t^2-1}{t^2-g^2}\right)^{\frac{1}{2}} dt \tag{A 1}$$

with cuts in the lower half ζ plane, where the square roots are positive when t is real and greater than g (> 1). The real constants g and h are to be chosen such that the fluid region is mapped on to the upper half ζ plane and the points $z = 0, ia, 2b+ia$ and $2b$ are mapped on to the respective points $\zeta = -g, -1, +1$ and $+g$. Thus g and h are defined implicitly, in terms of a and b, by the relations

$$b/h = \int_0^1 \left(\frac{1-t^2}{g^2-t^2}\right)^{\frac{1}{2}} dt, \quad a/h = \int_1^g \left(\frac{t^2-1}{g^2-t^2}\right)^{\frac{1}{2}} dt. \tag{A 2}$$

These integrals can be reduced to elliptic integrals K and E (see Gradshteyn & Rhyzhik 1965, for example) by the respective substitutions $t = \sin\theta$ and

$$t = \{g^2 - (g^2-1)\sin^2\theta\}^{\frac{1}{2}}.$$

Thus

$$b/(gh) = E(\mu) - \mu'^2 K(\mu), \quad a/(gh) = E(\mu') - \mu^2 K(\mu'), \tag{A 3}$$

where

$$\mu = 1/g, \quad \mu' = (1-\mu^2)^{\frac{1}{2}}. \tag{A 4}$$

Now the potential ϕ_0 has to vanish on the free surface (hence on the surface $\zeta = \xi$, $|\xi| > g$) while it has zero normal derivative on the scatterer ($\zeta = \xi, |\xi| < g$). From (2.5) the singular behaviour near $z = ia$ $(\zeta = -1)$ is given by

$$\begin{aligned}\phi_0 &\sim -\operatorname{Re}(z-ia)^{-\frac{2}{3}} \\ &\sim -\{9(g^2-1)/8h^2\}^{\frac{1}{3}} \operatorname{Re}(\zeta+1)^{-1}\end{aligned}$$

from the transformation (A 1) near $z = ia$. The solution in terms of ζ is therefore seen to be

$$\phi_0 = \left(\frac{9}{8h^2}\right)^{\frac{1}{3}} (g^2-1)^{-\frac{1}{6}} \operatorname{Re}\left\{\frac{i(\zeta^2-g^2)^{\frac{1}{2}}}{\zeta+1}\right\}, \tag{A}$$

which is formally exact, although ζ, g and h are given only implicitly in terms of and b by (A 1)–(A 3).

Our primary interest is to find the constant q that is related to the behaviour of ϕ_0 [formula (2.6)] near $z = 2b + ia$, where

$$\zeta - 1 \sim \{9(g^2-1)^{\frac{1}{2}}/8h^2\}^{\frac{1}{3}}(z - ia - 2b)^{\frac{2}{3}} e^{\frac{1}{3}i\pi}. \tag{A 6}$$

It follows from (A 5) and (2.6) that q is given exactly by

$$q = \tfrac{1}{16}(3/h)^{\frac{4}{3}}(g^2+1)(g^2-1)^{-\frac{1}{3}}, \tag{A 7}$$

with g and h defined by (A 3).

On eliminating h from (A 3), we find that $g = 1/\mu$ is the solution of the transcendental equation

$$a\{E(\mu) - \mu'^2 K(\mu)\} = b\{E(\mu') - \mu^2 K(\mu')\}, \tag{A 8}$$

which does not seem to be soluble for general values of a/b. Equation (A 8) can be simplified in a few special cases, as is now shown. If $a = b$, we have $\mu = \mu' = 1/\surd 2$, thus $g = \surd 2$ and h is then given by (A 3). The elliptic integrals can be expressed in terms of $\Gamma(\frac{1}{4})$ (Gradshteyn & Rhyzik 1965) to get

$$h = a2^{-\frac{1}{2}}\pi^{-\frac{3}{2}}\{\Gamma(\tfrac{1}{4})\}^2.$$

Hence

$$q = \frac{3\pi^2}{16}\left\{\frac{3\surd 2}{a\Gamma(\frac{1}{4})^2}\right\}^{\frac{4}{3}}. \tag{A 9}$$

If $b \ll a$, then $\mu \to 0$ and the elliptic integrals in (A 8) simplify to give

$$g^2 \sim \tfrac{1}{4}\pi a/b, \quad h^2 \sim 4ab/\pi.$$

Thus

$$q \sim \tfrac{1}{16}(3\pi/4b)^{\frac{4}{3}} \quad \text{as} \quad b/a \to 0. \tag{A 10}$$

Finally, if $a \ll b$ then $\mu' \to 0$ and we have

$$h \sim b, \quad g^2 \sim 1 + \frac{4a}{\pi b}, \quad q \sim \frac{a}{2\pi b}\left(\frac{3\pi}{4a}\right)^{\frac{4}{3}} \qquad \text{as} \quad a/b \to 0. \tag{A 11}$$

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